DEBYE INSTITUTE
UTRECHT UNIVERSITY

Partial exam "Toy Models"

Thursday, June 14th, 2018
Duration of the exam: max 3 hours

1. *Write your name and student number on all sheets*
2. *Write clearly, unreadable work cannot be corrected.*
3. *Max credits in between [ ]; total 100.*

Part 1: random walk

1. The so-called Brownian Ratchet model is a toy model that addresses how molecular machines produce directed motion. The model has been inspired by proteins or ligands that take directional walks over biomolecules such as DNA or microtubules. The idea of the Brownian Ratchet model is illustrated in Figure 1. In the Brownian

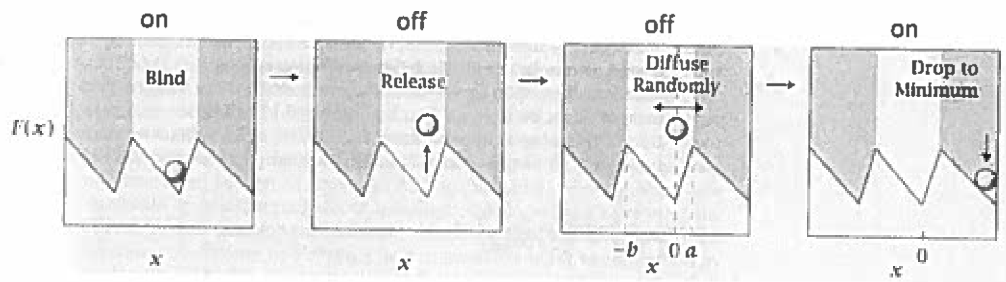

Figure 1: Brownian ratchet model

Ratchet model, we have an asymmetric, 1-dimensional periodic potential $F(x)$ in a coordinate x. The potential also is periodic in time, that is, it is 'on' during a time τ_{on} and it is 'off' during a time τ_{off}. During the time that the potential is turned off, the particle diffuses freely along the line. Once the potential is switched on again, the particle will return to its initial position if during the off-time $-b < x < a$, by sliding along the potential until the local minimum is reached. Likewise, it will move to the next minimum on the right if during τ_{off}, $x \geq a$. It will move to the next minimum on the left if during τ_{off}, $x \leq -b$. At long time intervals τ_{off}, particles are expected to move over several minimum positions.

a) [15] The probability of a particle to move over a distance in between x and $x+dx$ in time t is given by the random walk result (in one dimension) $P(x,t)dx = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-x^2/(4Dt)\right)dx$. Here D is the diffusion coefficient. Show that the probability for a particle to move one minimum position to the right is given by $p_{\text{right}} = \frac{1}{2}\text{erfc}\left(a/(2\sqrt{D\tau_{off}})\right)$, and to move one position to the left is given by $p_{\text{left}} = \frac{1}{2}\text{erfc}\left(b/(2\sqrt{D\tau_{off}})\right)$. The function $\text{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}du$ is the complementary error function. The lengths a and b are positive values.

b) [10] Argue that the net velocity to the right is under appropriate conditions given by $v_{\text{right}} \approx \frac{a+b}{\tau_{on}+\tau_{off}}(p_{\text{right}} - p_{\text{left}})$. Identify the condition(s) where this approximation holds.

c) [15] The probabilities have been plotted in Fig. 2 as a function of $2\sqrt{D\tau_{off}}$ for values of $a = 1$ and $b = 5$ (in units of $2\sqrt{D\tau_{off}}$). Show that $p_{\text{right}} - p_{\text{left}}$ has a maximum at $\tau_{off} = (b^2 - a^2)/(4D\ln(b/a))$. Make use of the property $\frac{\partial \text{erfc}(z)}{\partial z} = (-2/\sqrt{\pi})e^{-z^2}$. Explain qualitatively why a maximum is to be expected (also if you have not been able to provide the proof of the maximum).

d) [10] Provide an expression for the probability that after a time interval τ_{off}, the particle remains in its initial minimum.

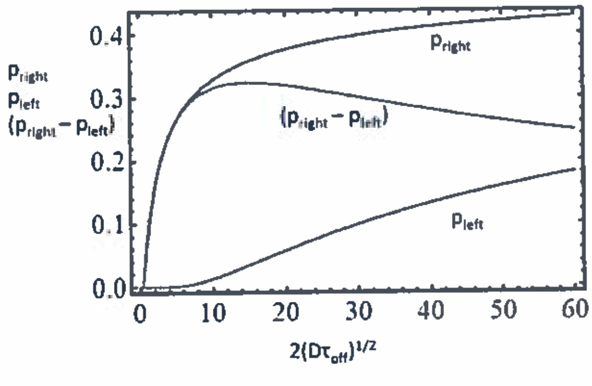

Figure 2: Displacement probabilities for lengths $a = 1$ and $b = 5$ as a function of $2\sqrt{D\tau_{off}}$

Part 2: Adsorption and MWC

2. A motor protein has two conformational states, M (Mobile) and S (Static). The ground state is the S state. The self energy of the motor protein in the S state is ϵ^S, and the self energy of the motor in the M state is ϵ^M. The motor protein

has two binding sites for inducer molecules. In the S state, the binding affinity for an inducer molecule is ϵ_S. In the M state, the affinity is ϵ_M. Be aware of the notation: superscrips S and M point to the motor self-energy, and subscripts S and M signify the binding strength of the motor binding sites to the inducers. Inducers are effectively ligands.

a) [10] Show that the grand partition function of a motor protein is given by $\Xi = \exp(-\beta\epsilon^S)(1+\lambda x_S)^2 + \exp(-\beta\epsilon^M)(1+\lambda x_M)^2$. Here $\beta = (k_BT)^{-1}$, $x_S = \exp(-\beta\epsilon_S)$, and $x_M = \exp(-\beta\epsilon_M)$. The grand partition function is defined as $\Xi = \sum_{n=0}^{n_{max}} \lambda^n Z(n, n_{max}, T)$, where $\lambda = e^{\beta\mu}$ with μ the chemical potential of an inducer molecule. Finally you may want to make use of the binomial theorem given by $\sum_n^{n_{max}} \binom{n_{max}}{n} p^N q^{N-n} = (p+q)^{n_{max}}$.

b) [10] Let u be defined as $u = \exp\left(-\beta(\epsilon^M - \epsilon^S)\right)$. Also, let $x_M = ax_S$, in other words, if $\epsilon_M = \epsilon_S + \Delta\epsilon$, then $a = \exp(-\beta\Delta\epsilon)$, so a reflects the difference in binding strength of the S and the M state with the inducers. Show that the probability of the motor in the M state can be written as

$$P_M = \frac{1}{1 + \frac{(1+\lambda x_S)^2}{u(1+a\lambda x_S)^2}}. \tag{1}$$

c) [10] Show that the fraction of occupied binding sites on the motor by inducer is given by

$$\theta = \frac{\langle n \rangle}{2} = \frac{\lambda x_S(1+\lambda x_S) + ua\lambda x_S(1+a\lambda x_S)}{(1+\lambda x_s)^2 + u(1+a\lambda x_s)^2}. \tag{2}$$

Here $\langle n \rangle$ is the average number of bound inducers to the motor protein.

d1) [10] The speed of the motor relative to its maximum speed v_{max} is given by $\frac{v}{v_{max}} = \theta\, P_M$. In a mutated version of the motor, it always is in the mobile state ($P_M = 1$) with affinity for inducers ϵ_M. In the mutant case, the relative speed of the motor is given by $\left(\frac{v}{v_{max}}\right)_{\text{mutant}} = \theta_{mutant}$. Show that $\left(\frac{v}{v_{max}}\right)_{\text{mutant}} = \frac{a\lambda x_s}{1+a\lambda x_s}$. (Hint: the derivation is similar to a-c).

d2) [10] In Fig. 3 the relative speed of the two-state motor protein and that of the mutant have been plotted as a function of λx_S, with values of $a = 10^4$ and $u = 10^{-2}$ (note that x_S and x_M are related via $x_M = ax_S$). Which of the two curves, A (left) or B (right) corresponds to the two-state motor, and which one to the mutant? There are at least two features of the curves that are significantly different and important. Identify these features and discuss the difference between the curves.

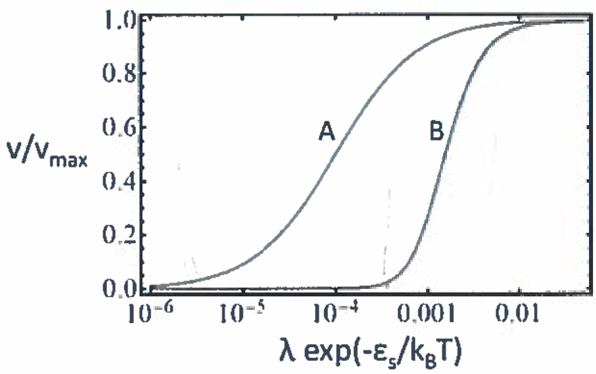

Figure 3: Relative speeds of a two-state motor protein and a mutant version with only a single state. Here $a = 10^4$, $u = 10^{-2}$.