Sketch of suggested solutions

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Problem 1.

a) The characteristic polynomial is $\det(A - \lambda \mathbb{I}) = \lambda^2 + 1$. The eigenvalues are the roots of the characteristic polynomial, hence $\lambda_{1/2} = \pm i$.

If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector associated with λ_1, it satisfies $Av = \lambda_1 v$ or equivalently solves the system of linear equations

$$\begin{aligned} (2-i)v_1 + v_2 &= 0 \\ -5v_1 - (2+i)v_2 &= 0. \end{aligned}$$

Hence,

$$v = c_1 \begin{pmatrix} -1 \\ 2-i \end{pmatrix}$$

for some $c_1 \in \mathbb{C}$.

Similarly, one obtains that $c_2 \begin{pmatrix} -1 \\ 2+i \end{pmatrix}$ with $c_2 \in \mathbb{C}$ are the eigenvectors corresponding to λ_2.

b) The matrix A is diagonalizable since the matrix

$$S = \begin{pmatrix} -1 & -1 \\ 2-i & 2+i \end{pmatrix},$$

whose columns consist of eigenvectors associated with λ_1 and λ_2, is invertible. Indeed, $\det S = -2i \neq 0$. One can show that then $A = SDS^{-1}$ with $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

c) The general solution to the homogeneous equation $\frac{d}{dt}F - AF = 0$ (or equivalently $\frac{d}{dt}F = AF$) is given by

$$F(t) = c_1 e^{it} \begin{pmatrix} -1 \\ 2-i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} -1 \\ 2+i \end{pmatrix} \quad \text{for } t \in \mathbb{R} \text{ with } c_1, c_2 \in \mathbb{C}.$$

We observe that the constant function $V_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $t \in \mathbb{R}$ is a particular solution to the inhomogeneous equation. The general solution is then given as the sum of this particular solution and the general solution to the homogeneous equation, i.e. $V = F + V_0$.

Problem 2.

a) By differentiating every term individually one finds that

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}$$

for $x \in \mathbb{R}$. Hence,

$$0 = f''(x) - \alpha f'(x) = \sum_{n=0}^{\infty} \big[(n+2)(n+1)a_{n+2} - \alpha(n+1)a_{n+1}\big] x^n,$$

and we infer from the identity principle that

$$\begin{aligned} (n+2)(n+1)a_{n+2} - \alpha(n+1)a_{n+1} &= 0 \qquad \text{or} \\ (n+2)a_{n+2} - \alpha a_{n+1} &= 0 \end{aligned} \tag{5}$$

for all integers $n \geq 0$, which is the sought recurrence relation.

b) After an index shift we get from (5) that

$$a_{n+1} = \frac{\alpha}{n+1} a_n$$

for $n \in \mathbb{N}$. Note that (5) does not impose any restriction on $a_0 \in \mathbb{R}$. Iterating the previous formula gives for integers $n \geq 0$,

$$a_{n+1} = \frac{\alpha^n}{(n+1)!} a_1.$$

Then,

$$f(x) = a_0 + \sum_{n=0}^{\infty} \frac{\alpha^n}{(n+1)!} a_1 x^{n+1} = a_0 + \sum_{n=1}^{\infty} \frac{a_1 \alpha^{n-1}}{n!} x^n, \qquad x \in \mathbb{R}.$$

The choice of a_0 and a_1 follows from the initial conditions. Indeed, $0 = f(0) = a_0$, and $1 = f'(0) = a_1$. All in all, we have that

$$f(x) = \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n!} x^n, \qquad x \in \mathbb{R}. \tag{6}$$

c) We will show that the series $\sum_{n\geq 1} b_{n,x}$ with $b_{n,x} = \frac{\alpha^{n-1}}{n!} x^n$ converges for any $x \in \mathbb{R}$. Since

$$\lim_{n\to\infty} \frac{|b_{n+1,x}|}{|b_{n,x}|} = \lim_{n\to\infty} \frac{|\alpha^n x^{n+1} n!|}{|(n+1)! x^n \alpha^{n-1}|} = \lim_{n\to\infty} \frac{|\alpha x|}{(n+1)} = 0,$$

the asserted convergence follows from the quotient test.

d) We observe that

$$\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n!} x^n = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha x)^n = \frac{1}{\alpha} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} (\alpha x)^n - 1 \Big),$$

which shows that (6) is the Taylor expansion of the function $x \mapsto \frac{1}{\alpha}(e^{\alpha x} - 1)$.

Problem 3.

a) The visualization of f is left to the reader. We observe that f is piecewise continuously differentiable with jumps in the derivatives for all odd integers, i.e. $x = 2k+1$ with $k \in \mathbb{Z}$.

b) For $k = 0$ we have that

$$\hat{f}_0 = \frac{1}{2} \int_{-1}^{1} f(x)\, dx = \int_0^1 1 - x^2\, dx = \frac{2}{3}.$$

We use the observation that $\hat{f}_k = \hat{f}_{-k}$ for $k \in \mathbb{Z}$ due to the fact that f is even along with the hint to obtain that

$$\begin{aligned}\hat{f}_k &= \frac{1}{2}(\hat{f}_k + \hat{f}_{-k}) = \frac{1}{4}\int_{-1}^{1}(1-x^2)(e^{-ik\pi x} + e^{ik\pi x})\,dx = \frac{1}{2}\int_{-1}^{1}(1-x^2)\cos(k\pi x)\,dx \\ &= -\int_0^1 x^2\cos(k\pi x)\,dx = \frac{-2(-1)^k}{\pi^2 k^2}.\end{aligned}$$

c) Since f is 2-periodic, piecewise continuously differentiable and continuous, we infer from the Fourier inversion formula that for $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2}(f(x^-) + f(x^+)) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{-ik\pi x},$$

where the series on the right-hand side converges. Since f is even, the Euler formula implies that

$$f(x) = \hat{f}_0 + 2\sum_{k=1}^{\infty} \hat{f}_k \cos(k\pi x),$$

which is the desired representation if we set $a_0 = \hat{f}_0$ and $a_k = 2\hat{f}_k$ for $k \in \mathbb{N}$ with the Fourier coefficients determined in b). Thus,

$$f(x) = \frac{2}{3} - 4\sum_{k=1}^{\infty} \frac{(-1)^k}{\pi^2 k^2}\cos(k\pi x). \tag{7}$$

d) For the first series we set $x = 0$ in (7) to find that

$$1 = f(0) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}.$$

This can be rewritten as

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}.$$

The Parseval formula tells us that $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}_k|^2$, where $\|\cdot\|$ is the norm induced by the standard inner product on $C^0_{2\text{-}per}((-1,1);\mathbb{C})$. Since

$$\|f\|^2 = \frac{1}{2}\int_{-1}^{1}(1-x^2)^2\,dx = \frac{8}{15}$$

and

$$\sum_{k=-\infty}^{\infty}|\hat{f}_k|^2 = |\hat{f}_0|^2 + 2\sum_{k=1}^{\infty}|\hat{f}_k|^2 = \frac{4}{9} + 2\sum_{k=1}^{\infty}\frac{4}{\pi^4 k^4} = \frac{4}{9} + \frac{8}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4},$$

it follows that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}.$$

Problem 4.

a) Observe that with $\mathcal{F}$ denoting the Fourier transformation with respect to the x-variable, we obtain for every $t > 0$ that

$$\mathcal{F}\big(\tfrac{\partial^2}{\partial x^2}u(\cdot,t)\big)(s) = (is)^2\hat{u}(s,t) = -s^2\hat{u}(s,t), \qquad s \in \mathbb{R},$$

and

$$\mathcal{F}\big(\tfrac{\partial^2}{\partial t^2}u(\cdot,t)\big)(s) = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2}u(x,t)e^{-isx}\,dx = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} u(x,t)e^{-isx}\,dx = \frac{\partial^2}{\partial t^2}\hat{u}(s,t), \qquad s \in \mathbb{R}.$$

Hence, applying $\mathcal{F}$ to (4) results in

$$\frac{\partial^2}{\partial t^2}\hat{u}(s,\cdot) = -s^2\hat{u}(s,\cdot)$$

for every $s \in \mathbb{R}$, or in other words, $\hat{u}(s,\cdot)$ solves the second order linear ordinary differential equation $v'' + s^2 v = 0$ in $(0,\infty)$.

b) If $s \neq 0$ the general (complex) solution to (4) is

$$v(t) = c_1 e^{ist} + c_2 e^{-ist}$$

with constants $c_1, c_2 \in \mathbb{C}$. We choose the constants such that the initial conditions are satisfied. It follows from

$$\hat{g}(s) = v(0) = c_1 + c_2 \qquad \text{and} \qquad 0 = v'(0) = is(c_1 - c_2)$$

that $c_1 = c_2 = \frac{1}{2}\hat{g}(s)$. Hence, $v(t) = \frac{1}{2}\hat{g}(s)(e^{ist} + e^{-ist}) = \hat{g}(s)\cos(st)$ for $t \geq 0$.

For $s = 0$ we know that every solution to (4) is of the form $v(t) = c_1 t + c_2$ with $c_1, c_2 \in \mathbb{C}$. Accounting for the initial conditions

$$\hat{g}(0) = v(0) = c_2 \qquad \text{and} \qquad 0 = v'(0) = c_1$$

implies that v is the constant function with value $\hat{g}(0)$.

This shows that for all $s \in \mathbb{R}$ the sought solution is $v(t) = \hat{g}(s)\cos(st)$ for $t \geq 0$.

c) For $s \in \mathbb{R}$, it holds that

$$(\mathcal{F}g_r)(s) = \int_{-\infty}^{\infty} g(r+x)e^{-isx}\,dx = \int_{-\infty}^{\infty} g(y)e^{-is(y-r)}\,dy = e^{isr}\int_{-\infty}^{\infty} g(y)e^{-isy}\,dy = e^{isr}\hat{g}(s),$$

where we have used the change of variables $y = r + x$.

d) In view of b) we have

$$\hat{u}(s,t) = \hat{g}(s)\cos(st) = \tfrac{1}{2}\hat{g}(s)(e^{ist} + e^{-ist})$$

for $s \in \mathbb{R}$ and $t \geq 0$. By Fourier inversion one obtains for every $t \geq 0$ that

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}(\hat{u}(\cdot,t))(x) = \tfrac{1}{2}\mathcal{F}^{-1}(\hat{g}(s)e^{ist})(x) + \tfrac{1}{2}\mathcal{F}^{-1}(\hat{g}(s)e^{-ist})(x) \\ &= \tfrac{1}{2}\mathcal{F}^{-1}(\mathcal{F}(g_t))(x) + \tfrac{1}{2}\mathcal{F}^{-1}(\mathcal{F}g_{-t})(x) = \tfrac{1}{2}g_t + \tfrac{1}{2}g_{-t} = \tfrac{1}{2}g(x+t) + \tfrac{1}{2}g(x-t), \qquad x \in \mathbb{R}. \end{aligned}$$

In the third equality, we have used the result from c). It is left to the reader to double-check that u is actually a solution to (2) and (3).

Problem 5.

a) Let $f, \tilde{f} \in C^0([0,1])$ and $\alpha \in \mathbb{R}$. Then for all $x \in \mathbb{R}$,

$$\begin{aligned} L_g(f+\tilde{f})(x) &= (\textstyle\int_0^1 f(y)+\tilde{f}(y)\,dy)g(x) = (\int_0^1 f(y)\,dy)g(x) + (\int_0^1 \tilde{f}(y)\,dy)g(x) \\ &= L_g(f)(x) + L_g(\tilde{f})(x) \end{aligned}$$

and

$$L_g(\alpha f)(x) = (\textstyle\int_0^1 \alpha f(y)\,dy)g(x) = \alpha(\int_0^1 f(y)\,dy)g(x) = \alpha L_g(f)(x).$$

This shows that $L_g(f+\tilde{f}) = L_g(f) + L_g(\tilde{f})$ and $L_g(\alpha f) = \alpha L_g(f)$. Hence, the operator L_g is linear.

b) Recall that λ is an eigenvalue for L_g if there exists a non-zero function $f \in C^0([0,1])$ such that

$$L_g(f) = \lambda f.$$

We will now show that $\lambda_g := \int_0^1 g(y)\,dy \neq 0$ is an eigenvalue of L_g. Indeed, for $f = g$ we find that

$$L_g(g) = \Big(\int_0^1 g(y)\,dy\Big)g = \lambda_g g.$$

Since g is not the zero function due to the assumption $\|g\| = 1$, this proves the assertion.

The corresponding eigenspace is

$$E_{\lambda_g} = \{f \in C^0([0,1]) : L_g(f) = \lambda_g f\} = \{f \in C^0([0,1]) : f = \alpha g \text{ for some } \alpha \in \mathbb{R}\} = \operatorname{span}\{g\}.$$

Indeed, since g is an eigenfunction of L_g for the eigenvalue λ_g, it is clear that $\operatorname{span}\{g\}$ has to be contained in E_{λ_g}. On the other hand, to see that E_{λ_g} cannot be larger, we observe that the condition $L_g f = \lambda_g f$ for any $f \in C^0([0,1])$ implies that $f = (\int_0^1 f(y)\,dy)\lambda_g^{-1} g$. Hence, f has to be a multiple of g.

c) In view of the condition $\|g\| = 1$, we calculate that

$$\begin{aligned} \langle L_g(f), g\rangle &= \int_0^1 (L_g(f))(x)g(x)\,dx = \int_0^1 (\textstyle\int_0^1 f(y)\,dy)g(x)g(x)\,dx \\ &= (\textstyle\int_0^1 f(y)\,dy)\displaystyle\int_0^1 g(x)g(x)\,dx = (\textstyle\int_0^1 f(y)\,dy)\langle g, g\rangle = (\int_0^1 f(y)\,dy)\|g\|^2 = \displaystyle\int_0^1 f(y)\,dy. \end{aligned}$$

This means that $L_g(f)$ and g are orthogonal if and only if f has vanishing mean value, i.e. $\int_0^1 f(y)\,dy = 0$.

d) Since $\int_0^1 f(y)\,dy \in \mathbb{R}$, we observe that $L_g(f)$ is a real multiple of g and hence an element of $\operatorname{span}\{g\}$. Therefore, we find that the orthogonal projection of $L_g(f)$ onto $\operatorname{span}\{g\}$ is again $L_g(f)$ and the distance is zero.