Wiskunde 3

SK-BWIS3

Universiteit Utrecht
Blok 3, 2017/18
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Final Exam

Name:

Student number:

Date: Wednesday, April 11, 2018

Time: 9:00 - 12:00 (3 hours)

Room: OLYMPOS, HAL3

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- Use a *separate* sheet for each problem.
- The use of textbooks, notes, calculators, cell phones, etc. is *not* allowed.
- Make sure that your answers are *readable* and *understandable.*
- Problems marked with * are bonus questions.

Total points: 48 (including bonus points)

Score:

1	2	3	4	5	Σ

Grade:

Problem 1.

Let the 2×2 matrix

$$A = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}$$

be given.

a) Determine the eigenvalues and eigenvectors of A. 3p

b) Show that A is diagonalizable by finding an invertible matrix $S \in \mathbb{C}^{2\times 2}$ and a diagonal matrix $D \in \mathbb{C}^{2\times 2}$ such that $A = SDS^{-1}$. *Hint:* It is not required to calculate the expression SDS^{-1} explicitly, but check that S is indeed invertible. 2p

c) Use the results from a) and b) to determine the general solution to the inhomogeneous system

$$\frac{d}{dt}V(t) - AV(t) = b$$

for $V : \mathbb{R} \to \mathbb{C}^2$, where A is as in a) and $b = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$. 3p

Problem 2.

Let $\alpha > 0$. Use the power series approach to find the solution to the differential equation

$$f'' - \alpha f' = 0 \qquad \text{in } \mathbb{R} \tag{1}$$

that satisfies $f(0) = 0$ and $f'(0) = 1$.

a) Plug the ansatz

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{with } a_n \in \mathbb{R}$$

into (1) and derive a recurrence relation for the coefficients a_n. 2p

b) Solve the recurrence relation from a) and use your findings to state an explicit power series formula (in dependence of α) for the desired solution. 3p

c) Show with the help of a convergence test of your choice that the power series in b) converges for all $x \in \mathbb{R}$. 2p

d)* Express the solution obtained in b) in terms of an exponential function. 2p

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Problem 3.

Let $f : \mathbb{R} \to \mathbb{C}$ be the 2-periodic function defined by

$$f(x) = 1 - x^2, \qquad x \in [-1, 1).$$

a) Visualize the graph of the function f by drawing at least two periods. 1p

b) Determine the Fourier coefficients $\hat{f}_k$ for $k \in \mathbb{Z}$. *Hint:* Taking account of the symmetry of f can simplify the calculation. You may use the integration formula

$$\int x^2 \cos(k\pi x)\, dx = \frac{2x\cos(k\pi x)}{\pi^2 k^2} - \frac{2\sin(k\pi x)}{\pi^2 k^3} + \frac{x^2 \sin(k\pi x)}{\pi k} + C \quad \text{for } k \in \mathbb{Z} \setminus \{0\}.$$

4p

c) Argue why f can be expressed as a converging Fourier cosine series, i.e.

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(k\pi x) \qquad \text{for } x \in \mathbb{R}.$$

In view of b), what are the coefficients a_k? 2p

d) Find the values of the two converging series

$$\sum_{k\geq 1} \frac{(-1)^k}{k^2} \qquad \text{and} \qquad \sum_{k\geq 1} \frac{1}{k^4}.$$

Hint: Use the findings from c), as well as Parseval's formula. 4p

Problem 4.

Consider the one-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \qquad (x,t) \in \mathbb{R} \times (0, \infty), \tag{2}$$

subject to the initial conditions

$$u(x,0) = g(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = 0 \qquad \text{for } x \in \mathbb{R}, \tag{3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function that is twice continuously differentiable such that g, g' and g'' are absolutely integrable.

a) Apply Fourier transformation to the equation (2) and show that for every $s \in \mathbb{R}$ the function $\hat{u}(s, \cdot)$ solves the ordinary differential equation

$$v'' + s^2 v = 0 \qquad \text{in } (0, \infty). \tag{4}$$

3p

b) For every $s \in \mathbb{R}$ determine the solution to (4) that satisfies the initial conditions $v(0) = \hat{g}(s)$ and $v'(0) = 0$. *Hint:* Treat the cases $s \neq 0$ and $s = 0$ separately. 3p

c)* For $r \in \mathbb{R}$, let $g_r(x) = g(x + r)$ for all $x \in \mathbb{R}$. Show that

$$(\mathcal{F} g_r)(s) = \hat{g}(s) e^{isr} \qquad \text{for } s \in \mathbb{R}.$$

Hint: Use a suitable change of variables. 2p

d) Use the results from b) and c) along with a Fourier inversion argument to solve (2) and (3). 3p

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Problem 5.

Recall that $C^0([0,1])$ denotes the space of continuous real-valued functions on the interval $[0,1]$. We endow $C^0([0,1])$ with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ and denote the induced norm by $\|\bullet\|$.
Moreover, let $g \in C^0([0,1])$ be a given function that satisfies $\|g\| = 1$ and $\int_0^1 g(x)\,dx \neq 0$.
In this question, we consider the operator

$$L_g : C^0([0,1]) \to C^0([0,1]), \qquad f \mapsto \Big(\int_0^1 f(y)\,dy\Big) g.$$

a) Prove that L_g is linear. 3p

b) Show that $\lambda_g = \int_0^1 g(y)\,dy$ is an eigenvalue of L_g and determine the corresponding eigenspace. 3p

c) Let $f \in C^0([0,1])$ be given. Calculate the inner product $\langle L_g(f), g\rangle$. Under what condition on f are $L_g(f)$ and g orthogonal? 2p

d)* For a given $f \in C^0([0,1])$, determine the orthogonal projection of $L_g(f)$ onto $\text{span}\{g\}$. What is the distance between $L_g(f)$ and $\text{span}\{g\}$? *Hint:* You may assume without proof that $\text{span}\{g\}$ is a linear subspace of $C^0([0,1])$. 1p